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Bounds and Exact Values of the Rainbow-k Domination Number for Selected Graph Families

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Gdańsk Conference on Graph Theory 2026

Talk plan

- Rainbow-k domination definition.
- General bounds for regular graphs.
- Exact values for cycles and Möbius ladders.
- An algorithm for $P(6t,t)$ and following conjectures.

This work has been supported by the European Commission's Horizon Europe Research and Innovation programme through the Marie Skłodowska-Curie Actions Staff Exchanges (MSCA-SE) under Grant Agreement no. 101182819 (COVER: *(C)ombinatorial (O)ptimization for (V)ersatile Applications to (E)merging u(R)ban Problems*).

Definition

Let G be a finite simple graph. A rainbow-k dominating function is

$$f : V(G) \to \{\emptyset, \{1\}, \dots, \{k\}\}$$

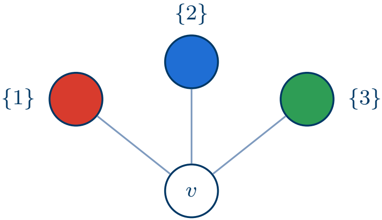
with the property that every uncolored vertex sees all k colors:

$$f(v) = \emptyset \implies \bigcup_{u \in N(v)} f(u) = \{1, \dots, k\}.$$

Its weight and the corresponding parameter are:

$$w(f) = |\{v \in V(G) : f(v) \neq \emptyset\}|,$$

$$\widetilde{\gamma}_k(G) = \min_f w(f).$$



If $f(v) = \emptyset$, then the colors seen by v are all of $\{1,2,3\}$:

$$\bigcup_{u \in N(v)} f(u) = \{1, 2, 3\}.$$

The two rainbow parameters

Classical k-rainbow domination [1]

One vertex can be assigned several colors:

$$f(v) \subseteq \{1, \ldots, k\}.$$

Rainbow-k domination

Every colored vertex can be assigned exactly one color:

$$f(v) \in \{\emptyset, \{1\}, \ldots, \{k\}\}.$$

- This makes the parameter more restrictive.
- It models coverage when one location can host only one resource type.
- The restriction creates rigid modular patterns in structured graphs.
- Rainbow-k domination was called *singleton t-rainbow domination* in some works.

Regular graph bounds

For an r-regular graph G of order n, $1 \le k \le r$:

Degree lower bound

$$\widetilde{\gamma}_k(G) \ge \left\lceil \frac{kn}{k+r} \right\rceil .$$

Sparse induced-subgraph lower bound

If $\alpha_s(G)$ is the maximum order of an induced subgraph with maximum degree at most s, then $\widetilde{\gamma}_k(G) \ge n - \alpha_{r-k}(G)$.

Probabilistic upper bound

$$\widetilde{\gamma}_k(G) \le n \cdot \min\left\{1, \frac{k(\ln r + 1)}{r}\right\}.$$

Related background on k-rainbow domination in regular graphs can be found in [3]; the probabilistic bound uses an alteration argument in the spirit of [4,5].

Degree lower bound

Let $C = \{v : f(v) \neq \emptyset\}$ be the set of colored vertices.

Comparing color supply and demand

Each uncolored vertex needs *all* k colors in its neighborhood, so the total color demand is at least

$$k(n - |C|).$$

Each colored vertex can supply its single color to at most r uncolored neighbors, so the total supply is at most

$$r|C|.$$

Hence

$$k(n - |C|) \leq r|C|.$$

Rearranging gives

$$|C| \geq \frac{kn}{k+r} \implies \widetilde{\gamma}_k(G) \geq \left\lceil \frac{kn}{k+r} \right\rceil.$$

Induced-subgraph lower bound

Let $U = V(G) \setminus C$ be the set of uncolored vertices.

Key observation

If $u \in U$, then among the r neighbors of u, at least k neighbors must be colored – one for each color. Therefore each uncolored vertex has at most

$$r - k$$

uncolored neighbors.

So the induced subgraph $G[U]$ has maximum degree at most $r - k$. By the definition of $\alpha_{r-k}(G)$,

$$|U| \leq \alpha_{r-k}(G).$$

Therefore

$$\widetilde{\gamma}_k(G) = |C| = n - |U| \geq n - \alpha_{r-k}(G).$$

Probabilistic upper bound I

Let G be r-regular. Fix $p \in [0,1]$. Independently for every vertex v, choose

$$f_0(v) = \begin{cases} \{i\}, & \text{with probability } p/k \text{ for each } i \in \{1,\dots,k\}, \\ \emptyset, & \text{with probability } 1-p. \end{cases}$$

Thus the expected number of initially colored vertices is

$$\mathbb{E}|C_0| = pn.$$

When is a vertex bad?

An uncolored vertex v is bad if some color is missing from its neighborhood. For a fixed color i,

$$\Pr(\text{color } i \text{ is absent from } N(v)) = \left(1 - \frac{p}{k}\right)^r \le e^{-pr/k}.$$

By the union bound,

$$\Pr(v \text{ is bad}) \le k\left(1 - \frac{p}{k}\right)^r \le ke^{-pr/k}.$$

Probabilistic upper bound II

Now repair the function by coloring every bad vertex arbitrarily. Then every vertex that remains uncolored sees all k colors, so the result is a valid rainbow-k dominating function. Hence

$$\mathbb{E}\, w(f) \leq n\left(p + ke^{-pr/k}\right).$$

Choosing $p = \dfrac{k \ln r}{r}$ gives

$$\mathbb{E}\, w(f) \leq n \cdot \frac{k(\ln r + 1)}{r}.$$

Together with the trivial bound $\widetilde{\gamma}_k(G) \leq n$, this yields

$$\widetilde{\gamma}_k(G) \leq n \cdot \min\left\{1, \frac{k(\ln r + 1)}{r}\right\}.$$

The $\frac{n}{2}$ extremal case

For an r-regular graph,

$$\widetilde{\gamma}_r(G) \geq \frac{n}{2}.$$

Equality structure

Equality holds exactly when $V(G)$ has a bipartition (U, C) and

$$C = C_1 \dot\cup C_2 \dot\cup \cdots \dot\cup C_r$$

such that for every $u \in U$,

$$|N(u) \cap C_i| = 1 \qquad (i = 1, \ldots, r).$$

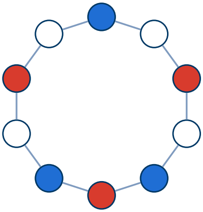
- For cubic graphs, equality forces $6 \mid n$ and three equally sized color classes.
- This is the structural reason why the later formulas depend on congruence classes.

Warm-up: cycles

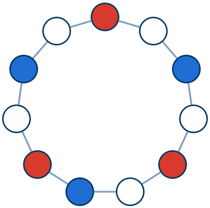
For every $n \geq 3$,

$$\widetilde{\gamma}_k(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & k = 1, \\ \frac{n}{2}, & k = 2,\ n \equiv 0 \pmod 4, \\ \left\lceil \frac{n+1}{2} \right\rceil, & k = 2,\ n \not\equiv 0 \pmod 4, \\ n, & k \geq 3. \end{cases}$$

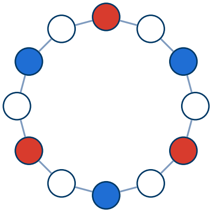
The case $k = 1$ is the ordinary domination number of a cycle [2].



$\widetilde{\gamma}_2(C_{10}) = 6.$



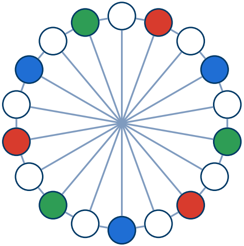
$\widetilde{\gamma}_2(C_{11}) = 6.$



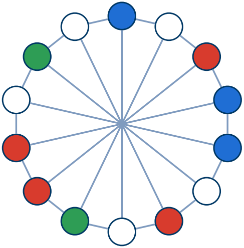
$\widetilde{\gamma}_2(C_{12}) = 6.$

Möbius ladders

The Möbius ladder M_n is obtained from a cycle C_n by adding opposite chords $v_i v_{i+n/2}$.



M_{18}: an optimal rainbow-3 dominating function of weight $9 = \frac{n}{2}$.



M_{14}: an optimal rainbow-3 dominating function of weight $9 = \frac{n}{2} + 2$.

Exact value for $\widetilde{\gamma}_3(M_n)$

For every even $n \geq 4$,

$$\widetilde{\gamma}_3(M_n) = \begin{cases} \frac{n}{2}, & n \equiv 6 \pmod{12}, \\ \frac{n}{2} + 1, & n \equiv 0, 4, 8 \pmod{12}, \\ \frac{n}{2} + 2, & n \equiv 2, 10 \pmod{12}. \end{cases}$$

Proof idea

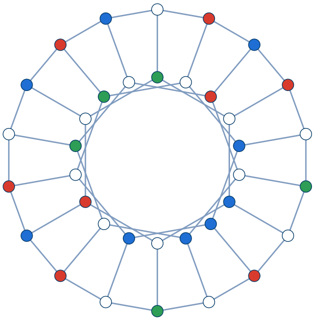
- The cubic lower bound already gives $\widetilde{\gamma}_3(M_n) \geq \frac{n}{2}$.
- Equality would force the rigid bipartite structure from the previous slide.
- The Möbius chord parity allows equality exactly for $n \equiv 6 \pmod{12}$.
- Other residues need one or two additional colored vertices.

Generalized Petersen graphs

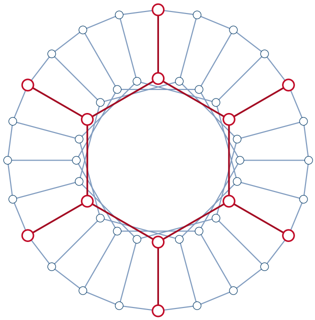
The generalized Petersen graph $P(d, e)$ has

$$V = \{u_0, \dots, u_{d-1}, v_0, \dots, v_{d-1}\},$$

with outer edges $u_i u_{i+1}$, spokes $u_i v_i$, and inner edges $v_i v_{i+e}$.



$P(18, 3)$ with an optimal rainbow-3 dominating function of weight 22.



$P(24, 4)$ with one block highlighted.

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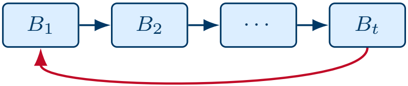
The family $P(6t, t)$

For $P(6t, t)$:

- the graph has $12t$ vertices;
- outer edges form a cycle;
- inner edges connect vertices with the same row index in consecutive blocks;
- grouping indices modulo t gives a cyclic sequence of six-row blocks.



Each block contributes one local state; the DP walks through the blocks cyclically.

DP step 0a: what is a state?

For fixed number of colors k, the DP state records only the six *outer* vertices of one block:

$$\sigma_j = (\sigma_{j,0}, \ldots, \sigma_{j,5}) \in \Sigma_k, \qquad \Sigma_k = \{0, 1, \ldots, k\}^6.$$

Here 0 means that the corresponding outer vertex is uncolored.

outer state σ_j:

0	1	0	2	0	1

These six entries correspond to the six outer vertices $u_j^0, \ldots, u_j^5$.

What is not stored?

The six inner vertices of the block are not part of the global DP state. Their colors are chosen locally in the precomputation step.

Thus the number of states is $S = |\Sigma_k| = (k+1)^6$.

DP step 0b: the local table

For three consecutive outer states

$$\alpha, \beta, \gamma \in \Sigma_k,$$

the algorithm precomputes the best possible coloring of the six inner vertices of the middle block.

Local cost table

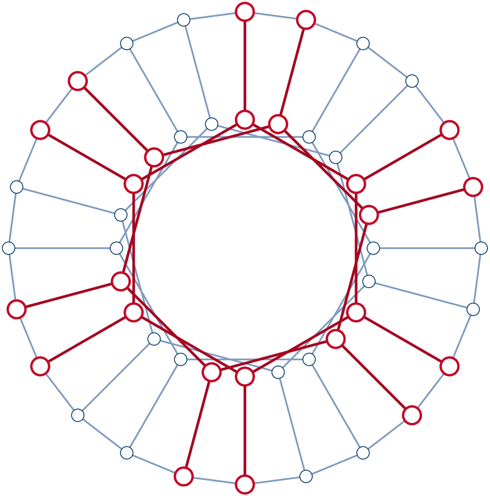
$$\mu_k(\alpha, \beta, \gamma) = \min_{\tau \in \Sigma_k} \; c(\tau),$$

where τ ranges over feasible colorings of the six inner vertices of the middle block.

- α, β, γ fix the outer context around the middle block.
- τ is used only to compute μ_k; it is not stored in the DP table.
- After this precomputation, transitions can use μ_k as a local cost.

DP step 1: start

We illustrate the procedure on $P(24,4)$, i.e. $t=4$.



Red rings and red edges show the blocks introduced so far.

Initial pair

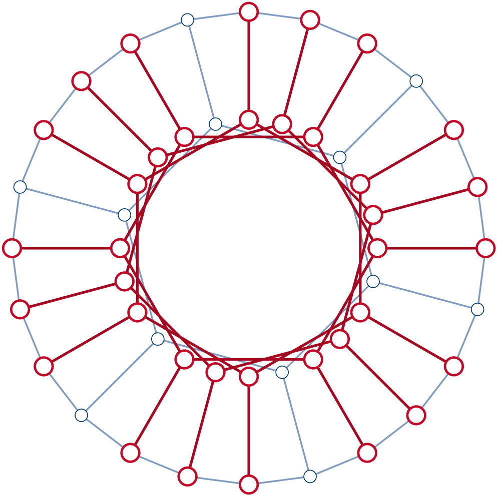
The DP starts by fixing two consecutive outer states

$$\sigma_0, \sigma_1 \in \Sigma_k.$$

$$D_1(\sigma_0, \sigma_1) = c(\sigma_0) + c(\sigma_1).$$

- The state stores only outer vertices.
- Inner vertices are handled later by the local table μ_k.

DP step 2: extend



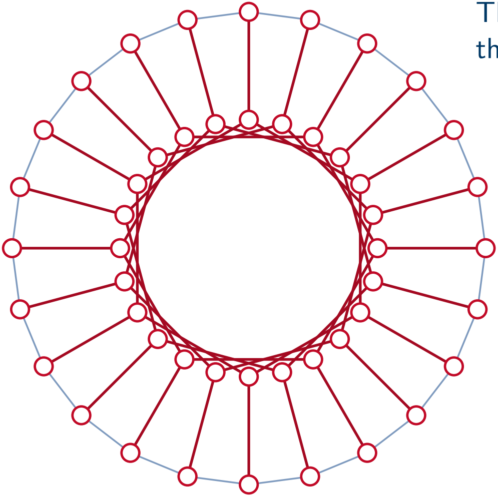
For each target pair (β, γ), we minimize over the possible previous outer state α:

$$D_{j+1}(\beta,\gamma) = \min_{\alpha \in \Sigma_k} \Big(D_j(\alpha,\beta) + c(\gamma) + \mu_k(\alpha,\beta,\gamma) \Big).$$

Meaning

$c(\gamma)$ pays for the new outer block. $\mu_k(\alpha,\beta,\gamma)$ pays for the best inner coloring of the middle block.

DP step 3: repeat



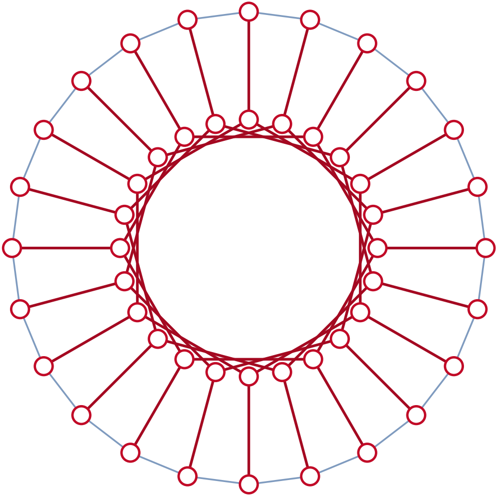
The transition is repeated around the cyclic sequence of blocks:

$$(\sigma_0, \sigma_1) \to (\sigma_1, \sigma_2) \to \cdots$$

$$\to (\sigma_{t-2}, \sigma_{t-1}).$$

- In $P(24, 4)$ there are four DP blocks.
- For general $P(6t, t)$, there are t blocks.
- The table keeps the best value for each pair of last outer states.

DP step 4: close the cycle



For every choice of the initial outer states σ_0, σ_1 and the final outer states α, β, we add the two remaining local costs needed to close the cycle. Then we take the minimum over all four boundary states:

$$\mathrm{OPT} = \min_{\sigma_0,\sigma_1,\alpha,\beta\in\Sigma_k} \Big(D_{t-1}^{\sigma_0,\sigma_1}(\alpha,\beta) + \mu_k(\alpha,\beta,\rho^+(\sigma_0)) + \mu_k(\rho^-(\beta),\sigma_0,\sigma_1) \Big).$$

Running time

If $S = (k+1)^6$, precomputation takes $O(S^4)$, and the DP takes $O(tS^5)$, hence $O(nS^5)$.

Conjectures for $P(6t,t)$

Two colors

For $t \geq 2$,

$$\widetilde{\gamma}_2(P(6t,t)) = \begin{cases} 18, & t = 3, \\ 5t + 1 + \left\lfloor \frac{t}{8} \right\rfloor + \left\lfloor \frac{t+5}{8} \right\rfloor, & t \neq 3. \end{cases}$$

Three colors

For $t \geq 2$,

$$\widetilde{\gamma}_3(P(6t,t)) = \begin{cases} 22, & t = 3, \\ 6t, & t \equiv 1,5 \pmod 6, \\ 6t+3, & t \equiv 0,2,4 \pmod 6, \\ 6t+6, & t \equiv 3 \pmod 6,\ t \geq 9. \end{cases}$$

These formulas are suggested by the DP computations and refine the picture known from generalized Petersen rainbow-domination results [6–8].

Conclusion

- Rainbow-k domination is locally simple: every uncolored vertex must see all k colors in its neighborhood.
- This, however, makes the problem globally rigid in structured graphs.
- Exact values for cycles and Möbius ladders show strong modular behaviour.
- For $P(6t, t)$, a finite-state DP gives exact computations and suggests periodic conjectures.

Next steps

Proving the Petersen conjectures, extending the block-DP approach to $P(ct, t)$, and studying further regular graph families, including 4- and 5-regular graphs.

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