Bounds and Exact Values of the Rainbow-k Domination Number for Selected Graph Families

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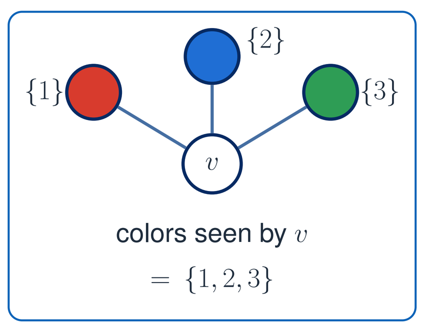
Piotr Lange • Gdańsk University of Technology
Faculty of Electronics, Telecommunications and Informatics • piotr.lange@student.pg.edu.pl

A simple local rule – every uncolored vertex must see all k colors – produces rigid global behaviour on structured graph families.

1. Rainbow-k domination



Let G be a finite simple graph and let $k \geq 1$. We define a **rainbow-k dominating function** as

$$f : V(G) \longrightarrow \{\emptyset, \{1\}, \{2\}, \dots, \{k\}\}$$

such that every uncolored vertex sees all colors:

$$f(v) = \emptyset \quad \Longrightarrow \quad \bigcup_{u \in N(v)} f(u) = \{1, 2, \dots, k\}.$$

Other names: Rainbow-k domination was called *singleton t-rainbow domination* in some works.
Relation to k-rainbow domination: In rainbow-k domination, each colored vertex carries *exactly one* color. This differs from classical k-rainbow domination [1], where one vertex may carry several colors.

The weight is the number of colored vertices,

$$w(f) = |\{v \in V(G) : f(v) \neq \emptyset\}|,$$

and the parameter studied here is

$$\tilde{\gamma}_k(G) = \min_f w(f).$$

● $\{1\}$ ● $\{2\}$ ● $\{3\}$ ○ $\emptyset$

2. General bounds for regular graphs

For an r-regular graph G of order n and $1 \leq k \leq r$:

Degree lower bound

$$\tilde{\gamma}_k(G) \geq \left\lceil \frac{kn}{k+r} \right\rceil.$$

Sparse induced-subgraph bound

Let $\alpha_s(G)$ be the maximum order of an induced subgraph of maximum degree at most s. Then

$$\tilde{\gamma}_k(G) \geq n - \alpha_{r-k}(G).$$

Probabilistic upper bound

$$\tilde{\gamma}_k(G) \leq n \cdot \min\left\{1, \frac{k(\ln r + 1)}{r}\right\}.$$

The first bound is local and degree-based, in the spirit of regular rainbow-domination estimates [3]. The second also records how large the uncolored induced subgraph may be. The upper bound comes from a probabilistic alteration argument [4,5].

3. The $\frac{n}{2}$ extremal case

For an r-regular graph G of order n, the lower bound gives $\tilde{\gamma}_r(G) \geq \frac{n}{2}$.

Equality characterization

$$\tilde{\gamma}_r(G) = \frac{n}{2}$$

if and only if $V(G)$ has a bipartition (U, C) and

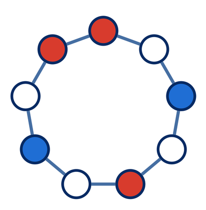
$$C = C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_r$$

such that every uncolored-side vertex sees exactly one neighbor of each color:

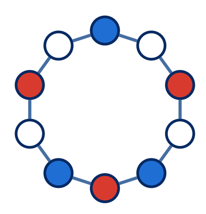
$$|N(u) \cap C_i| = 1 \quad (u \in U,\ i = 1, \dots, r).$$

For cubic graphs this forces $6 \mid n$ and three equally sized color classes. This rigidity is the structural reason why the presented formulas depend on congruence classes.

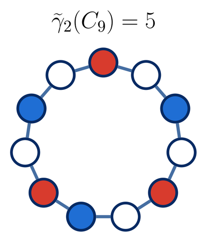
4. Warm-up: cycles



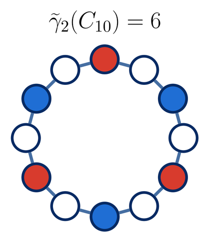
$\tilde{\gamma}_2(C_9) = 5$



$\tilde{\gamma}_2(C_{10}) = 6$



$\tilde{\gamma}_2(C_{11}) = 6$



$\tilde{\gamma}_2(C_{12}) = 6$

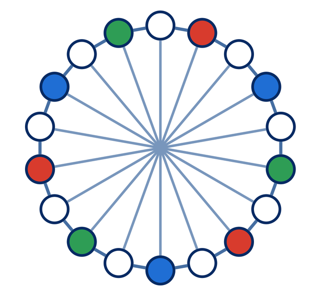
For every cycle C_n, $n \geq 3$, with the case $k = 1$ corresponding to ordinary domination [2],

$$\tilde{\gamma}_k(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & k = 1, \\ \frac{n}{2}, & k = 2,\ n \equiv 0 \pmod 4, \\ \left\lceil \frac{n+1}{2} \right\rceil, & k = 2,\ n \not\equiv 0 \pmod 4, \\ n, & k \geq 3. \end{cases}$$

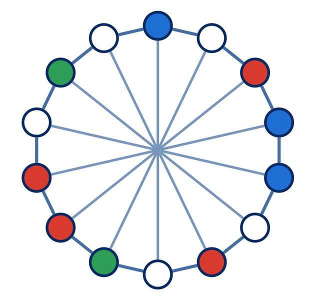
The four drawings show the $k = 2$ pattern around the transition from odd cycles to the half-colorable case C_{12}.

5. An exact family: Möbius ladders

The Möbius ladder M_n is obtained from a cycle by adding opposite chords $v_i v_{i+n/2}$.



$n \equiv 6 \pmod{12}$: $\tilde{\gamma}_3(M_{18}) = 9 = n/2$



$n \equiv 2 \pmod{12}$: $\tilde{\gamma}_3(M_{14}) = 9 = n/2 + 2$

Exact formula for $\tilde{\gamma}_3(M_n)$

For every even $n \geq 4$,

$$\tilde{\gamma}_3(M_n) = \begin{cases} \frac{n}{2}, & n \equiv 6 \pmod{12}, \\ \frac{n}{2} + 1, & n \equiv 0, 4, 8 \pmod{12}, \\ \frac{n}{2} + 2, & n \equiv 2, 10 \pmod{12}. \end{cases}$$

Why modulo 12 appears

In a cubic graph, $\tilde{\gamma}_3(G) \geq \lceil \frac{n}{2} \rceil$. Equality is rigid: the graph must admit a bipartition (U, C) and a partition

$$C = C_1 \dot{\cup} C_2 \dot{\cup} C_3$$

such that every $u \in U$ has exactly one neighbor in each C_i.
For M_n, this occurs only when the ladder-chord parity and divisibility conditions fit together:

$$n \equiv 6 \pmod{12}.$$

Equality construction: If $n = 12d + 6$, color all odd vertices and leave all even vertices uncolored:

$$f(v_i) = \begin{cases} \emptyset, & i \equiv 0 \pmod 2, \\ \{1\}, & i \equiv 1 \pmod 6, \\ \{2\}, & i \equiv 3 \pmod 6, \\ \{3\}, & i \equiv 5 \pmod 6. \end{cases}$$

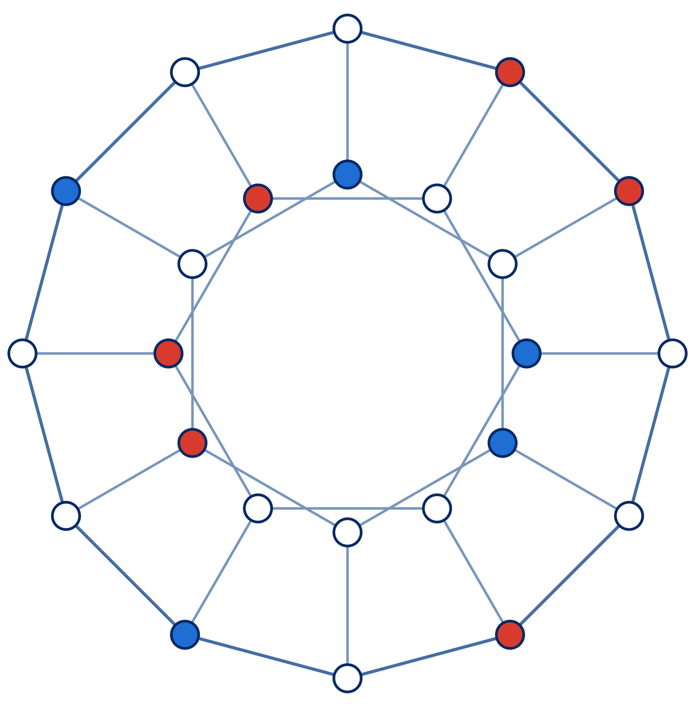
The case M_{14} shows the $n \equiv 2 \pmod{12}$ residue, where two extra colored vertices are needed.

6. Generalized Petersen graphs $P(6t, t)$

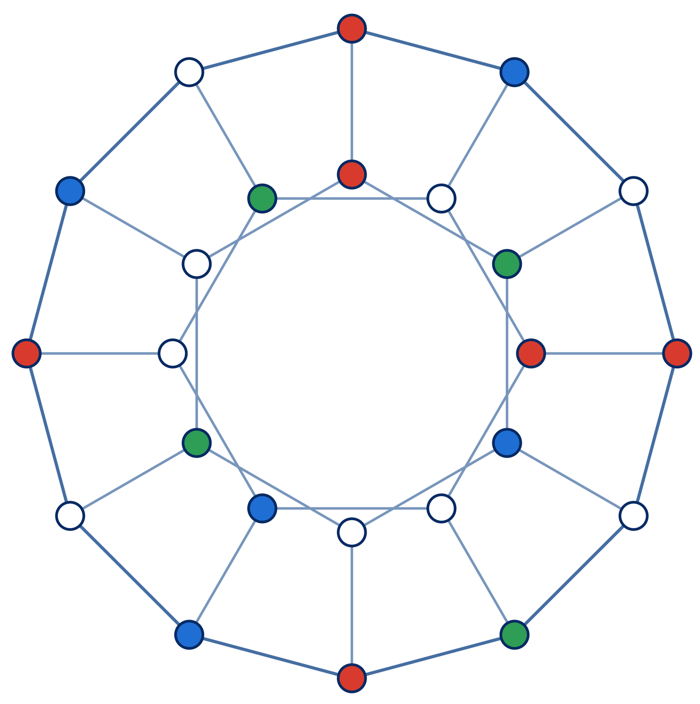
Recall that a generalized Petersen graph $P(d, e)$ has

$$V = \{u_0, \dots, u_{d-1}, v_0, \dots, v_{d-1}\},$$

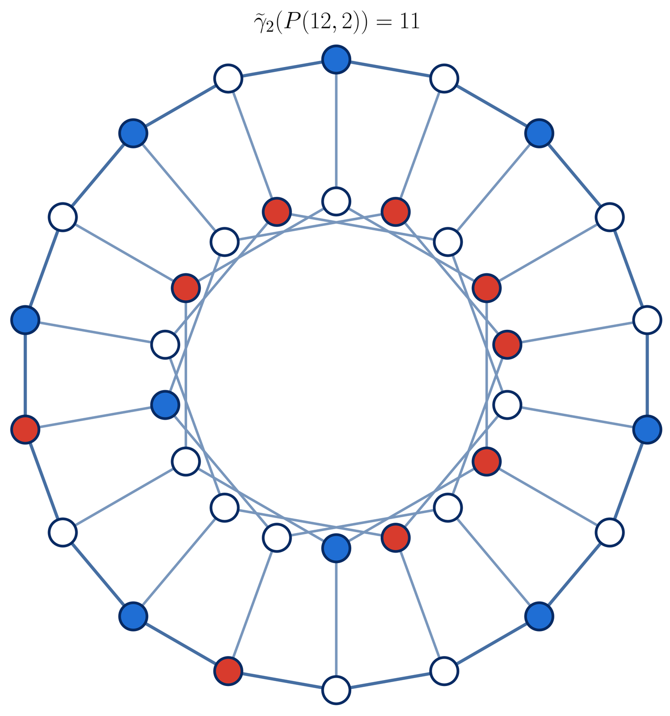
with outer edges $u_i u_{i+1}$, spokes $u_i v_i$, and inner edges $v_i v_{i+e}$.
Previous results on rainbow domination in generalized Petersen graphs provide the baseline for our exact values and conjectures [6–8].



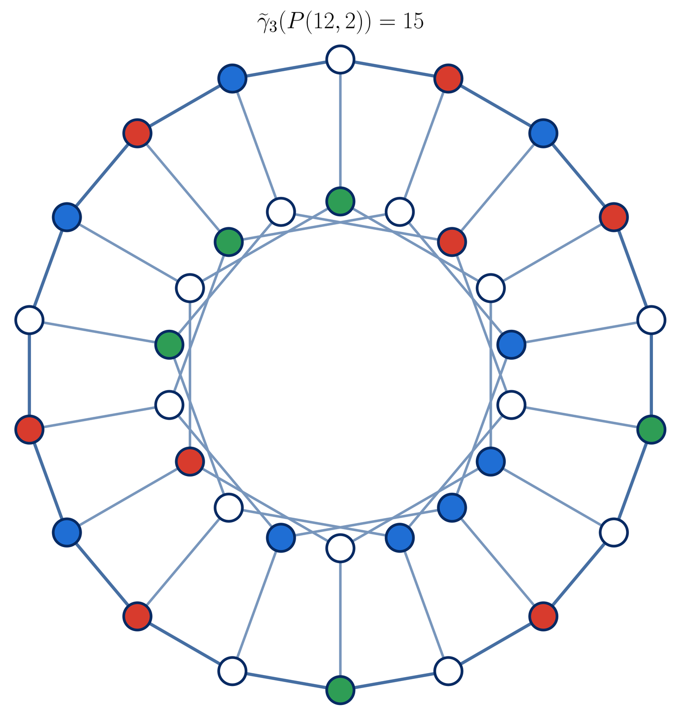
$\tilde{\gamma}_2(P(12, 2)) = 11$



$\tilde{\gamma}_3(P(12, 2)) = 15$



$\tilde{\gamma}_2(P(18, 3)) = 18$



$\tilde{\gamma}_3(P(18, 3)) = 22$

For $P(6t, t)$, the six-row block structure gives an exact finite-state dynamic program.

six-row states → local table μ_k → cyclic block DP → exact value

If k is the number of colors and $S = (k+1)^6$, the dynamic program runs in

$$O(n \cdot S^5),$$

so for every fixed k it is linear in the number of vertices.

7. Computations and conjectures

The exact dynamic program is practical for $k = 2$. For $k = 3$, evidence was obtained by integer linear programming. The conjectures below sharpen the known generalized Petersen results [6–8].

Conjecture for two colors

For every integer $t \geq 2$,

$$\tilde{\gamma}_2(P(6t, t)) = \begin{cases} 18, & t = 3, \\ 5t + 1 + \left\lceil \frac{t}{8} \right\rceil + \left\lceil \frac{t+5}{8} \right\rceil, & t \neq 3. \end{cases}$$

Conjecture for three colors

For every integer $t \geq 2$,

$$\tilde{\gamma}_3(P(6t, t)) = \begin{cases} 22, & t = 3, \\ 6t, & t \equiv 1, 5 \pmod 6, \\ 6t + 3, & t \equiv 0, 2, 4 \pmod 6, \\ 6t + 6, & t \equiv 3 \pmod 6 \text{ and } t \geq 9. \end{cases}$$

8. Open problems

- Proving the conjectural formulas for $P(6t, t)$.
- Extending the block-DP approach to broader generalized Petersen families $P(ct, t)$.
- Studying other graph families, including 4- and 5-regular graphs.

9. Conclusion

Rainbow-k domination is locally simple but globally rigid.
The restriction that each vertex may host at most one color turns neighborhood coverage into a strong structural constraint. On cycles, Möbius ladders, and generalized Petersen graphs, this produces exact formulas, modular phenomena, and specialized dynamic programming algorithms.

Acknowledgement

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